Calculation of the charge density on a grounded and ungrounded conducting sphere near a point charge $+q$.

Description:

This calculation shows examples of calculating charge densities on spherical objects in an external electric field provided by a point charge. Whilst grounded objects are commonly solved in examples, ungrounded objects can also be handled as the net charge neutrality takes the place of a boundary condition and allows the problem to be solved.

Intended Audience, Prerequisites:

Level 200, 300 University.
Vector Calculus and a first course in Electromagnetism.

Keywords:

Laplace's equation, Laplacian, Gauss's Law, Legendre Polynomials.

1) Charge density on a <u>grounded</u> conducting sphere near a point charge $+q$.

To solve this, we need the first of Maxwell's equations, that is Gauss's law for electrostatics, which when written in terms of the potential function φ (ie: $\vec{E}(\bar{r}) = -\nabla\varphi$) gives Poisson's equation. As we will be dealing with the field outside of the point charge and conductors, we can take $\rho(\bar{r}) = 0$ and then simply solve Laplace's equation.

$$\boxed{\nabla \circ \vec{E}(\bar{r}) = \frac{\rho(\bar{r})}{\epsilon_0}} \qquad \text{(Gauss's Law)}$$

$$\boxed{\nabla^2\varphi(\bar{r}) = \frac{-\rho(\bar{r})}{\epsilon_0}} \qquad \text{(Poisson's Eqn.)}$$

$$\boxed{\nabla^2\varphi(\bar{r}) = 0} \qquad \text{(Laplace's Eqn.)}$$

As our starting point, if the charge distribution is azimuthally symmetric the solution to Laplace's equation is an expansion of the Legendre polynomials, we will not be deriving this here, but starting from this point.

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} (a_l r^l + \frac{b_l}{r^{l+1}}) P_l(\cos\theta) \tag{1}$$

where a_l, b_l are to be determined from the boundary conditions and the physical requirement that φ is finite throughout space (or square integrable). As an example, for a radius outside of the sphere $r > R$ we need only consider the b_l as a_l must vanish for the potential φ to be square integrable (ie: tend to zero as r tends to ∞).

The usual course of action in solving these problems to is to expand the field supplied by the point charge $+q$ (or any other charge distribution for that matter) along the (positive) axis of symmetry, and then cast it into a form similar to that of the Legendre expansion shown above for $\varphi(r,\theta)$.

So for a charge $+q$ placed at a distance d on the z-axis we can expand the potential with the help of the binomial theorem noting $r/d < 1$,

$$\begin{aligned}\varphi_{(+q)}(R \le r < d, z \text{ axis}) &= k_e \frac{q}{(d-r)} = k_e \frac{q/d}{(1-r/d)} = k_e \frac{q}{d}(1 + (\frac{r}{d}) + (\frac{r}{d})^2 + ...)\\ &= k_e q \sum_l \frac{r^l}{d^{l+1}}\end{aligned}$$

which can now be recognised in a form similar to (1), with $k_e q/d^{l+1}$ taking on the role of the a_l coefficients present in the Legendre expansion. As the potential of the point charge has aximuthal symmetry about the z-axis (the axis along we just performed the expansion) we can now simply multiply this by the $P_l(\cos\theta)$ to get the general form for angle's lying off the z-axis.

$$\varphi_{(+q)}(R \le r < d, \theta) = k_e q \sum_l \frac{r^l}{d^{l+1}} P_l(\cos\theta)$$

We now make the claim that the charge distribution present on the grounded sphere will also be azimuthally symmetric, for instance if you were to rotate the sphere or charge by any angle around the z-axis the problem would appear identical, and therefore there could not be any variation azimuthally in the charge distribution that is induced on the grounded sphere. So that for the grounded sphere, with consideration of (1), we will have a contribution of;

$$\varphi_{sphere}(R \le r, \theta) = \sum_l \frac{b_l}{r^{l+1}} P_l(\cos\theta)$$

Where the b_l are what we need to find to solve this problem. By the principle of superposition for electric fields, the total electric field in the region $(R < r < d, \theta)$ will be the sum of φ_{+q} and φ_{sphere}.

$$\varphi(R \le r < d, \theta) = \sum_{l=0}^{\infty} (k_e q \frac{r^l}{d^{l+1}} + \frac{b_l}{r^{l+1}}) P_l(\cos\theta) \tag{2}$$

In electrostatics the electric field inside a conductor is always identically zero,

$$\vec{E}(r < R, \theta) = 0$$

The sphere is grounded so we can apply this as a boundary condition,

$$\varphi(R,\theta) = 0$$
$$0 = \sum_{l=0}^{\infty} (k_e q \frac{R^l}{d^{l+1}} + \frac{b_l}{R^{l+1}}) P_l(\cos\theta)$$

which can only be true if each individual coefficient of P_l vanishes.

$$0 = k_e q \frac{R^l}{d^{l+1}} + \frac{b_l}{R^{l+1}}$$
$$b_l = -k_e q \frac{R^{2l+1}}{d^{l+1}}$$

So we have as our complete solution on the interval $(R \leq r < d)$, inserting into (2) (and neglecting to write our interval for r from hence forth),

$$\varphi(r,\theta) = k_e q \sum_{l=0}^{\infty} (\frac{r^l}{d^{l+1}} - \frac{R^{2l+1}}{d^{l+1} r^{l+1}}) P_l(\cos\theta) \tag{3}$$

To now find the surface charge density we need to apply Gauss's law in integral form

$$\boxed{\oint \vec{E}(\bar{r}) \circ d\bar{A} = \frac{q_{enc.}}{\epsilon_0}} \tag{Gauss's Law}$$

We form a Gaussian region, with a surface as close to the (outer) surface of the sphere as infinitesimal calculus will allow, and with the other sides extending in the interior of the sphere. The surfaces on the interior of the sphere will not contribute to the integral, as $E(r < R, \theta)$ is identically equal to zero on the inside of the sphere (whether the sphere is hollow or solid makes no difference). The only surface of consequence in the integral is the one at (or infinitesimally close to) the spherical surface.

The ∇ operator in spherical polar is given by,

$$\nabla = \frac{\partial}{\partial r}\hat{\text{r}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}$$

We need only concern ourselves with the radial component, as our Gaussian surface just outside of the sphere has a radial orientation. All other components of the electric field vanish in any case when $r = R$.

$$\vec{E}_r = -\frac{\partial}{\partial r}\varphi(r,\theta)$$
$$= -k_e q \sum_{l=0}^{\infty} (\frac{l r^{l-1}}{d^{l+1}} + \frac{(l+1)R^{2l+1}}{d^{l+1} r^{l+2}}) P_l(\cos\theta)$$

Applying Gauss's Law in integral form over the one contributing surface,

$$dA\frac{\sigma(R,\theta)}{\epsilon_0} = -k_e q \sum_{l=0}^{\infty} (\frac{l r^{l-1}}{d^{l+1}} + \frac{(l+1)R^{2l+1}}{d^{l+1} r^{l+2}}) P_l(\cos\theta)|_{r=R}\hat{\text{r}} \circ dA\hat{\text{r}}$$
$$\frac{\sigma(R,\theta)}{\epsilon_0} = -k_e q \sum_{l=0}^{\infty} (2l+1)\frac{R^{l-1}}{d^{l+1}} P_l(\cos\theta)$$
$$\sigma(R,\theta) = -\frac{q}{4\pi} \sum_{l=0}^{\infty} (2l+1)\frac{R^{l-1}}{d^{l+1}} P_l(\cos\theta)$$

As one might expect the charge distribution on the sphere is negative, the positive charge $+q$ has induced some negative charge to be stored on the sphere, this negative charge is freely available from the ground connection. There is a net negative charge on the sphere. The precise charge density can be evaluated at the poles owing to the fact that $P_l(\cos 0) = P_l(1) \equiv 1$ and $P_l(-1) = (-1)^l P_l(1)$. These properties of the Legendre polynomials will not be developed here but are taken for granted. The infinite sum that is then remaining can be cast into a hypergeometric form and from which a computer algebra system will be necessary to find it's closed form. This is left as an exercise for the reader.
For example the north pole yields:

$$\frac{\sigma(R,\theta=0)}{\epsilon_0} = -\frac{k_e q}{Rd}\frac{(1+R/d)}{(1-R/d)^2}$$

In addition the total charge on the sphere can be found by integrating the charge density and using another property of the Legendre polynomials, $\int_{-1}^{1} P_l(x)\,dx = 0,\ \forall\ l \neq 0$

2) Charge density on an ungrounded conducting sphere near a point charge $+q$.

In this case, the problem progresses much like the preceding example. The main difference is we don't have a boundary condition to directly apply to the sphere, however we can indirectly get around this problem. The sphere will be at a constant potential, as all conductor's are, and therefore we can solve for a dummy potential V_0 under the condition that the integral of the surface charge density vanishes (ie: the sphere is charge neutral). You probably have an instinct the answer should be either the potential of the center of the sphere or perhaps the average of the potential of a spherical surface in the radial field. They are slightly different and it would be hard to guess definitively.

Reproducing our original solution (2) we have on the interval $(R < r < d)$,

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} (k_e q \frac{r^l}{d^{l+1}} + \frac{b_l}{r^{l+1}}) P_l(\cos\theta)$$

Suppose that $\varphi(R,\theta) = V_0$, we don't know yet what V_0 is, unlike in the preceding example when this was equal to zero (ie: grounded means zero potential).

$$\varphi(R,\theta) = \sum_{l=0}^{\infty} (k_e q \frac{R^l}{d^{l+1}} + \frac{b_l}{R^{l+1}}) P_l(\cos\theta) = V_0 \tag{4}$$

We state without proof the orthogonality condition for Legendre polynomials for $x = \cos\theta$ (note: these properties are not being developed here but merely applied to the problem). As θ ranges from 0 to π, x ranges from 1 to -1. The Jacobian for spherical integration mops up the negative sign so the limits can be reversed, ie: $d(\cos\theta) = -\sin\theta\, d\theta$.

$$\int_{-1}^{1} P_l(x) P_m(x) = \frac{2}{2l+1} \delta_{lm}$$

Now we can act on (4) with $P_m(x)$ and integrate from -1 to 1 term by term, and make use of $\int_{-1}^{1} P_l(x)\, dx = 0,\ \forall\, l \neq 0$ for the left hand side.

$$\int_{-1}^{1} P_m(x) \cdot V_0\, dx = \int_{-1}^{1} \sum_{l=0}^{\infty} (k_e q \frac{R^l}{d^{l+1}} + \frac{b_l}{R^{l+1}}) P_m(x) \cdot P_l(x) dx$$

$$V_0 \left(\frac{2}{2m+1}\right) \delta_{m=0} = \sum_{l=0}^{\infty} (k_e q \frac{R^l}{d^{l+1}} + \frac{b_l}{R^{l+1}}) \left(\frac{2}{2l+1}\right) \delta_{l,m=0}$$

The LHS is generally zero except for when $m = 0$ as shown above, in this case we find

$$b_0 = R\left(V_0 - \frac{k_e q}{d}\right)$$

For all other $m \neq 0$ we have

$$b_m = -k_e q \frac{R^{2m+1}}{d^{m+1}}$$

Inserting these into (2) we obtain a full solution for the potential on the interval ($R < r < d$)

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} (k_e q \frac{r^l}{d^{l+1}}) P_l(\cos\theta) + \frac{R}{r}\left(V_0 - \frac{k_e q}{d}\right) + \sum_{l=1}^{\infty} (-k_e q \frac{R^{2l+1}}{r^{l+1} d^{l+1}}) P_l(\cos\theta)$$

And proceeding as before to find the radial derivative, and also explicitly taking care of summation indices on the first summation to allow a regrouping later on

$$\begin{aligned} \vec{E_r} &= -\frac{\partial}{\partial r}\varphi(r,\theta) \\ &= -\left\{ \sum_{l=1}^{\infty} k_e q (\frac{l r^{l-1}}{d^{l+1}}) P_l(\cos\theta) - \frac{R}{r^2}\left(V_0 - \frac{k_e q}{d}\right) + \sum_{l=1}^{\infty} (k_e q \frac{(l+1) R^{2l+1}}{r^{l+2} d^{l+1}}) P_l(\cos\theta) \right\} \end{aligned}$$

Let $r \to R$ so the radial electric field just outside the sphere is

$$\begin{aligned} \vec{E_r}\Big|_{r=R} &= -\left\{ \sum_{l=1}^{\infty} k_e q (\frac{l R^{l-1}}{d^{l+1}} + \frac{(l+1) R^{2l+1}}{R^{l+2} d^{l+1}}) P_l(\cos\theta) - \frac{R}{R^2}\left(V_0 - \frac{k_e q}{d}\right) \right\} \\ &= -\left\{ \sum_{l=1}^{\infty} k_e q (\frac{R^{l-1}}{d^{l+1}} (2l+1) P_l(\cos\theta) - \frac{V_0}{R} + \frac{k_e q}{R d} \right\} \\ &= -\left\{ \sum_{l=0}^{\infty} k_e q (\frac{R^{l-1}}{d^{l+1}} (2l+1) P_l(\cos\theta) - \frac{V_0}{R} \right\} \end{aligned}$$

Now we can employ Gauss's Law in integral form once again on the single surface contributing to the integral,

$$\begin{aligned} dA \frac{\sigma(R,\theta)}{\epsilon_0} &= -\left\{ \sum_{l=0}^{\infty} k_e q (\frac{R^{l-1}}{d^{l+1}} (2l+1) P_l(\cos\theta) - \frac{V_0}{R} \right\} \hat{\mathrm{r}} \circ dA \hat{\mathrm{r}} \\ \frac{\sigma(R,\theta)}{\epsilon_0} &= -\left\{ \sum_{l=0}^{\infty} k_e q (\frac{R^{l-1}}{d^{l+1}} (2l+1) P_l(\cos\theta) - \frac{V_0}{R} \right\} \end{aligned}$$

Given an expression for the charge density we are going to integrate over the entire spherical surface and set it equal to zero, as our ungrounded sphere must be charge neutral as there was no grounding wire available to have deposited charge on it. Our Jacobian for this surface integral is simply $2\pi R^2 \int_{-1}^{1} ...dx$, recall that we converted from θ to x with $x = \cos\theta$.

$$\begin{aligned} \int_{-1}^{1} \frac{\sigma(R,x)}{\epsilon_0} dx &= \int_{-1}^{1} \left\{ -\sum_{l=0}^{\infty} k_e q (\frac{R^{l-1}}{d^{l+1}} (2l+1) P_l(\cos\theta) + \frac{V_0}{R} \right\} dx \\ 0 &= \int_{-1}^{1} \left\{ -\sum_{l=0}^{\infty} k_e q (\frac{R^{l-1}}{d^{l+1}} (2l+1) P_l(\cos\theta) \right\} dx + \frac{2V_0}{R} \end{aligned}$$

Again making use of $\int_{-1}^{1} P_l(x)\, dx = 0,\ \forall\, l \neq 0$ we can pick out $P_0(x)$ only, where $P_0(x) \equiv 1$

$$\begin{aligned} 0 &= -\sum_{l=0}^{\infty} k_e q (\frac{R^{l-1}}{d^{l+1}} (2l+1) \cdot (2\delta_{l=0}) + \frac{2V_0}{R} \\ 0 &= \frac{2V_0}{R} - \frac{2k_e q}{R d} \\ V_0 &= \frac{k_e q}{d} \end{aligned}$$

Which concludes the problem, we find the potential of an ungrounded sphere in an external radial field is merely the potential of the position of the center of the sphere.