Cosine Academy

Calculation with Ampere's Law in integral form.

Description:

This calculation shows a simple application of Ampere's Law in integral form around a current carrying wire, one of the few example's that can directly be solved by hand. The integral is kept in symbolic form right until the end to help gain an apprecation of how these line integral's magically equal zero whenever the amperian loop is brought just outside of the current carrying elements.

Intended Audience, Prerequisites:

Level 200 University.
Vector Calculus and a first course in Electromagnetism.

Keywords:

Ampere's Law, Biot-Savart Law, Vector Calculus, Line integral, Hypergeometric function.

1) Magnetic field outside an infinitely long current carrying wire.

Beginning with the Biot-Savart law which gives the magnetic field at a position $\bar{r}$ from the current element $d\bar{l}$ for a uniform current

$$\boxed{d\vec{B} = \left(\frac{\mu_0}{4\pi}\right)\frac{I d\bar{l} \rtimes \bar{r}}{r^3}} \qquad \text{(Biot-Savart Law)}$$

Or in vector integral form (making the replacement of $\bar{r}$ with $\bar{s}$ instead, so that $\bar{s}$ denotes the source vector of the directed current elements from the Biot-Savart Law)

$$\vec{B}(\bar{r}) = \left(\frac{\mu_0 I}{4\pi}\right)\int_l \frac{d\bar{s} \rtimes (\bar{r} - \bar{s})}{|\bar{r} - \bar{s}|^3}$$

The parameterization of the wire will be along the x-axis so we can write $\bar{s} = x_s\mathbf{i}$, such that $d\bar{s} = dx_s\mathbf{i}$ with limits of $\pm\infty$, and $\bar{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\vec{B}(\bar{r}) = \left(\frac{\mu_0 I}{4\pi}\right)\int_{-\infty}^{\infty} \frac{dx_s\mathbf{i} \rtimes ((x - x_s)\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{((x - x_s)^2 + y^2 + z^2)^{3/2}}$$

where the cross product is found via:

$$dx_s\mathbf{i} \rtimes ((x - x_s)\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ dx_s & 0 & 0 \\ x - x_s & y & z \end{vmatrix} = 0\mathbf{i} - z.dx_s\mathbf{j} + y.dx_s\mathbf{k}$$

$$\vec{B}(\bar{r}) = \left(\frac{\mu_0 I}{4\pi}\right)\int_{-\infty}^{\infty} \frac{-z.dx_s\mathbf{j} + y.dx_s\mathbf{k}}{((x - x_s)^2 + y^2 + z^2)^{3/2}}$$

Make the substitution $\sqrt{y^2 + z^2}\tan\theta = x - x_s$, so $-\sqrt{y^2 + z^2}\sec^2\theta = dx_s$, limits are then respectively $-\pi/2, \pi/2$

$$\begin{aligned}
\vec{B}(\bar{r}) &= \left(\frac{\mu_0 I}{4\pi}\right)\int_{\pi/2}^{-\pi/2} \frac{\sqrt{y^2 + z^2}(+z.\sec^2\theta\, d\theta\,\mathbf{j} - y.\sec^2\theta\, d\theta\,\mathbf{k})}{(y^2 + z^2)^{3/2}(\tan^2\theta + 1)^{3/2}} \\
&= \left(\frac{\mu_0 I}{4\pi}\right)\frac{1}{(y^2 + z^2)}\int_{\pi/2}^{-\pi/2} \frac{\sec^2\theta}{\sec^3\theta}(z\mathbf{j} - y\mathbf{k})d\theta \\
&= \left(\frac{\mu_0 I}{4\pi}\right)\frac{1}{(y^2 + z^2)}\int_{\pi/2}^{-\pi/2} \cos\theta(z\mathbf{j} - y\mathbf{k})d\theta \\
&= \left(\frac{\mu_0 I}{4\pi}\right)\frac{1}{(y^2 + z^2)}(-2z\mathbf{j} + 2y\mathbf{k}) \\
&= \left(\frac{\mu_0 I}{2\pi}\right)\frac{-z\mathbf{j} + y\mathbf{k}}{(y^2 + z^2)}
\end{aligned}$$

2) Calculation of a circular *Amperian loop* around an infinitely long current carrying wire.

As previously determined the magnetic field outside an infinitely long current carrying wire lying along the x-axis is given by:

$$\vec{B}(\bar{r}) = \left(\frac{\mu_0 I}{2\pi}\right)\frac{-z\mathbf{j} + y\mathbf{k}}{(y^2 + z^2)}$$

An *amperian loop* in the yz plane can be represented as

$$r^2 = (y - y_0)^2 + z^2$$

which is a circle of radius r centered at $(0, y_0, 0)$.Without loss of generality, we can restrict to $y_0 \geq 0$ and if for instance the circle's origin was lying off the $z = 0$ plane then the coordinate system can always be re-oriented via a rotation to bring to the form of coordinates shown. This is possible because of cylindrical/solenoidal symmetry in the vector field $\vec{B}$. The x co-ordinate is non consequential as we can always slide the axis along the abstracted infinitely long wire without any material effect in the magnetic field.

Ampere's Circuital Law (in modern days just referred to as Ampere's Law) is

$$\boxed{\oint_C \vec{B} \circ d\bar{r} = \mu_0 I_{enc}} \qquad \text{(Ampere's Law)}$$

We are going to directly calculate the line integral in this worked example. The parameterization of C is given by $C : \bar{r} = (r\cos\theta + y_0)\mathbf{j} + r\sin\theta\mathbf{k}$ for $0 \leq \theta \leq 2\pi$, therefore $d\bar{r} = (-r\sin\theta\mathbf{j} + rcos\mathbf{k})\,d\theta$. Before proceeding I will note this is one of the very few problems in magnetism in which an exact solution is possible to arrive at by hand, it's not great that we had to approximate the wire as infinitely long in the first place, but that aside the following is very elegant in that it finally works out.

$$\begin{aligned}
\oint_C \vec{B} \circ d\bar{r} &= \oint \left(\frac{\mu_0 I}{2\pi}\right) \frac{-z\mathbf{j} + y\mathbf{k}}{(y^2+z^2)} \circ d\bar{r} \\
&= \left(\frac{\mu_0 I}{2\pi}\right) \int_0^{2\pi} \frac{(-r\sin\theta\mathbf{j} + (r\cos\theta + y_0)\mathbf{k}) \circ (-r\sin\theta\mathbf{j} + rcos\mathbf{k})}{r^2\sin^2\theta + (r\cos\theta + y_0)^2} d\theta \\
&= \left(\frac{\mu_0 I}{2\pi}\right) \int_0^{2\pi} \left(\frac{r^2 + y_0 r\cos\theta}{r^2 + y_0^2 + 2y_0 r\cos\theta}\right) d\theta
\end{aligned}$$

Wolfram Alpha will return a symbolic solution of

$$\frac{1}{2}\left(\theta - 2\arctan\left(\frac{(r+y_0)\cot(\frac{\theta}{2})}{r - y_0}\right)\right)$$

which is difficult to make use of at the limits, of course it can be man-handled into any integral multiple of π depending on what branch of the tan function is used. Instead of settling for this solution we will solve the integral exactly with a hypergeometric function as follows;

$$\begin{aligned}
\oint_C \vec{B} \circ d\bar{r} &= \left(\frac{\mu_0 I}{2\pi}\right) \int_0^{2\pi} \left(\frac{r^2 + y_0 r\cos\theta}{r^2 + y_0^2 + 2y_0 r\cos\theta}\right) d\theta \\
&= \left(\frac{\mu_0 I}{2\pi}\right) \frac{1}{2} \int_0^{2\pi} \left(\frac{2r^2 + 2y_0 r\cos\theta}{r^2 + y_0^2 + 2y_0 r\cos\theta}\right) d\theta \\
&= \left(\frac{\mu_0 I}{2\pi}\right) \frac{1}{2} \cdot \int_0^{2\pi} \left(\frac{r^2 + (r^2 + y_0^2 + 2y_0 r\cos\theta) - y_0^2}{r^2 + y_0^2 + 2y_0 r\cos\theta}\right) d\theta \\
&= \left(\frac{\mu_0 I}{2\pi}\right) \frac{1}{2} \cdot \int_0^{2\pi} \left(1 + \frac{r^2 - y_0^2}{r^2 + y_0^2 + 2y_0 r\cos\theta}\right) d\theta \\
&= \left(\frac{\mu_0 I}{2\pi}\right) \left\{\pi + \frac{1}{2}\left(\frac{r^2 - y_0^2}{r^2 + y_0^2}\right) \int_0^{2\pi} \left(\frac{1}{1 + \frac{2y_0 r}{r^2 + y_0^2}\cos\theta}\right) d\theta\right\}
\end{aligned}$$

Halfway there, we need another π to make it's way out of this. Let $k = \frac{2y_0 r}{r^2 + y_0^2}$, you can convince yourself that k is always less than 1 for any two positive numbers, and is exactly when 1 when $r = y_0$. In the case of $r = y_0$, the wire is exactly on the boundary chosen and the integral has convergence issues. This should not be of too much concern to ignore for in the first instance a real wire has some breadth and therefore a current density. It is worth stating that the idea of current and current density are an approximation of reality in any case, where in fact there are real discrete charge carriers (electrons) that occupy some volume in space due to their wave-function, so in classical electromagnetism current is treated as the infinitesimal representation of all these many charge carriers. Ignoring an infinitely thin boundary here then is no need for concern.

Focusing on the integral itself (which incidentally is common in physics and fairly difficult with elementary methods) we note the following peculiar results, the first two obvious from cosine having a periodicity over 2π. The last can be demonstrated via partial fractional decomposition.

$$\int_0^{2\pi} \left(\frac{1}{1 + k\cos\theta}\right) d\theta = \int_0^{2\pi} \left(\frac{1}{1 - k\cos\theta}\right) d\theta = \int_0^{2\pi} \left(\frac{1}{1 - k^2\cos^2\theta}\right) d\theta$$

We note that the middle form above can be expanded with the binomial theorem, and then we can make use of the limits to greatly simplify the problem.

$$\int_0^{2\pi} \left(\frac{1}{1 - k\cos\theta}\right) d\theta = \int_0^{2\pi} (1 + k\cos\theta + k^2\cos^2\theta + k^3\cos^3\theta + k^4\cos^4\theta + ...)d\theta$$

All the terms with odd powers of cosine vanish on the integration interval $[0, 2\pi]$. This leaves even terms only, which is what we would have arrived at starting with the 3rd representation of the integral above in any case.

$$\int_0^{2\pi} \left(\frac{1}{1 - k\cos\theta}\right) d\theta = \int_0^{2\pi} (1 + k^2\cos^2\theta + k^4\cos^4\theta + k^6\cos^6\theta + k^8\cos^8\theta + ...)d\theta$$

Using integral results for the cosine function we have;

$$\int \cos^n u\, du = \frac{1}{n}\cos^{n-1} u \cdot \sin u + \frac{(n-1)}{n} \int \cos^{n-2} u\, du, \int_0^{2\pi} \cos^2 u = \frac{u}{2} + \frac{1}{4}\sin 2u\Big]_0^{2\pi} = \pi$$

$$\text{noting} \int_0^{2\pi} \cos^n u \cdot \sin u\, du = 0, \forall n \in \mathbb{N}$$

So each time the reduction formula is applied the sin term vanishes. We can build up the following ladder,

$$\int_0^{2\pi} k^2 \cos^2\theta \, d\theta = (2\pi)\frac{1}{2}k^2$$

$$\int_0^{2\pi} k^4 \cos^4\theta \, d\theta = (2\pi)\frac{3\cdot 1}{4\cdot 2}k^4$$

$$\int_0^{2\pi} k^6 \cos^6\theta \, d\theta = (2\pi)\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}k^6$$

$$\int_0^{2\pi} k^8 \cos^8\theta \, d\theta = (2\pi)\frac{7\cdot 5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}k^8$$

$$\int_0^{2\pi} k^{10} \cos^{10}\theta \, d\theta = (2\pi)\frac{9\cdot 7\cdot 5\cdot 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2}k^{10}$$

Going back to our original problem we can write

$$\oint_C \vec{B}\circ d\bar{r} = \left(\frac{\mu_0 I}{2\pi}\right)\left\{\pi + \frac{1}{2}\left(\frac{r^2-y_0^2}{r^2+y_0^2}\right)\left(2\pi + 2\pi(\frac{1}{2}k^2 + \frac{3\cdot 1}{4\cdot 2}k^4 + \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}k^6 + ...)\right)\right\}$$

with $k = \frac{2y_0 r}{r^2+y_0^2}$, which is strictly a positive number in the way we have set up the problem. The series that resulted from the cosine integration can be put in the form of a hypergeometric series. Let's consider the series of,

$$\underbrace{1}_{\text{Term 0}} + \underbrace{\frac{1}{2}k^2}_{\text{Term 1}} + \underbrace{\frac{3\cdot 1}{4\cdot 2}k^4}_{\text{Term 2}} + \underbrace{\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}k^6}_{\text{Term 3}} + \underbrace{\frac{7\cdot 5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}k^8}_{\text{Term 4}}$$

which has the from $c_0 + c_1k^2 + c_2k^4 + c_3k^6 + c_4k^8 + ..., c_0 = 1$

where

$$\frac{c_{n+1}}{c_n} = \frac{(2n+1)}{(2n+2)}k^2 = \frac{(n+\frac{1}{2})}{(n+1)}k^2$$

Therefore we can write

$$(\frac{1}{2}k^2 + \frac{3\cdot 1}{4\cdot 2}k^4 + \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}k^6 + ...) = {}_1F_0(\frac{1}{2};k^2) - 1$$

$$\oint_C \vec{B}\circ d\bar{r} = \left(\frac{\mu_0 I}{2\pi}\right)\left\{\pi + \frac{1}{2}\left(\frac{r^2-y_0^2}{r^2+y_0^2}\right)\left(2\pi + 2\pi({}_1F_0(\frac{1}{2};k^2) - 1)\right)\right\}$$

You probably don't recognize the taylor series of the above hypergeometric function, it's simply

$${}_1F_0(\frac{1}{2};k^2) = \frac{1}{\sqrt{1-k^2}}$$

Noting that k was strictly positive and $0 \le k < 1$ so we are only dealing with the positive square root.

$$\begin{aligned}\sqrt{1-k^2} &= \frac{\sqrt{(r^2-y_0^2)^2}}{(r^2+y_0^2)} > 0\\ &= \frac{(r^2-y_0^2)}{(r^2+y_0^2)}, r > y_0\\ &= \frac{(y_0^2-r^2)}{(r^2+y_0^2)}, y_0 > r\end{aligned}$$

Collecting all the results

$$\begin{aligned}\oint_C \vec{B}\circ d\bar{r} &= \left(\frac{\mu_0 I}{2\pi}\right)\left\{\pi + \pi\left(\frac{r^2-y_0^2}{r^2+y_0^2}\right)\left(\frac{r^2+y_0^2}{r^2-y_0^2}\right)\right\}, r > y_0\\ &= \mu_0 I\end{aligned}$$

or when the loop does not contain the wire

$$\begin{aligned}\oint_C \vec{B}\circ d\bar{r} &= \left(\frac{\mu_0 I}{2\pi}\right)\left\{\pi - \pi\left(\frac{r^2-y_0^2}{r^2+y_0^2}\right)\left(\frac{r^2+y_0^2}{r^2-y_0^2}\right)\right\}, y_0 > r\\ &= 0\end{aligned}$$

Perhaps you are wondering what other shapes can be integrated by hand, well I can tell you if the circular loop is inclined to the yz plane so that it projects an an ellipse onto the yz plane then the integral is extremely difficult and is of the form

$$\frac{a + b\cos\theta}{c + d\cos\theta + e\cos^2\theta}$$

so already with just a minor change to the *amperian loop* the integral probably will have you heading to a computer algebra system.